# Summary on Functions

(1) Rule of f means the expression of f.

(2) Domain of f means the set of input values of f.
Range means the set of output values of f.
Domain and range must be presented in proper set notation.

For example: $R_f = [0, \infty)$. For this set, 0 is included.
$R_f = (2, 3)$ means 2 and 3 are not included in the set.

(3) Sketch the functions according to domain.

Your sketch should include the end-points, asymptotes and intersection with axes if any. Stationary points are included if it helps in finding the range of the function.

Caution: Do not just sketch the entire graph obtained using the GC.

| | **Inverse function $f^{-1}$** | **Composite function fg** |
|---|---|---|
| Exist? | Check 1 to 1. See # below on how to show f is 1 to 1. | Check $R_g \subseteq D_f$ |
| Rule | Let $y = f(x)$ and make $x$ the subject. See example 1 on the next page. | Replace $x$ in $f(x)$ by $g(x)$. See example 2 on the next page |
| Domain | Domain of $f^{-1}$ = Range of f | Domain of fg = Domain of g |
| Range | Range of $f^{-1}$ = Domain of f | 2 methods:<br>(1) Sketch the graph of $fg(x)$ based on the domain of fg and find range based on the graph.<br>(2) Use $R_g$ as domain of f and find the corresponding range of f using the graph of f<br>See example 3 on the next page. |
| What if the function does not exist? | Restrict to a subset of the domain of f so that f is 1 to 1 | Restrict $x$ to a subset of domain of g so that $R_g \subseteq D_f$ |

**# To show 1 to 1 function.**

Sketch the graph of f according to its domain and if it is 1 to 1, state that any horizontal line $y = k$ cuts the graph of the function **at most one point**, therefore it is f is 1 to 1.

## Relationships between f and $f^{-1}$ and the line $y = x$

- The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other about the line $y = x$.
- $ff^{-1}(x) = x, \quad x \in \text{Domain of } f^{-1}$ and $f^{-1}f(x) = x, \quad x \in \text{Domain of } f$.

  These results can be quoted without proof and are always true no matter what function f is.
  The functions have the same rule but their domains may differ.
  When asked to sketch $y = f^{-1}f(x)$, we should sketch the line $y = x$ where $x \in \text{Domain of } f$.

**Example 1:** Find an expression for the inverse of f where $f(x) = x^2 + x - 1, x \le -\frac{1}{2}$.

To make $x$ the subject for $y = x^2 + x - 1$.

**Method 1:** $x^2 + x - 1 - y = 0$ and use the formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to write $x$ in terms of $y$

$$x = \frac{-1 \pm \sqrt{1 - 4(-1 - y)}}{2} = \frac{-1 \pm \sqrt{5 + 4y}}{2}.$$

Reject one of the expression using the domain of f.

Since $x \le -\frac{1}{2}$, $x = \frac{-1 - \sqrt{5 + 4y}}{2}$.

Therefore, $f^{-1}(x) = \frac{-1 - \sqrt{5 + 4x}}{2}$.

**Method 2:** Complete the square for $x$.

$$y = x^2 + x - 1 = \left(x + \frac{1}{2}\right)^2 - \frac{5}{4} \Rightarrow \left(x + \frac{1}{2}\right)^2 = y + \frac{5}{4} \Rightarrow x = -\frac{1}{2} \pm \sqrt{y + \frac{5}{4}}$$

Since $x \le -\frac{1}{2}$, $x = -\frac{1}{2} - \sqrt{y + \frac{5}{4}}$.

Therefore, $f^{-1}(x) = -\frac{1}{2} - \sqrt{x + \frac{5}{4}}$.

**Example 2 Finding the composite function.**

Given that $f(x) = x^2 - 1, x \in \mathbb{R}$ and $g(x) = \ln x, x \in \mathbb{R}$.
Find the rule of fg($x$) and of gf($x$).

$$fg(x) = f(g(x)) = f(\ln x) = (\ln x)^2 - 1$$
$$gf(x) = g(f(x)) = g(x^2 - 1) = \ln(x^2 - 1)$$

**Example 3 (find the range of composite function)**

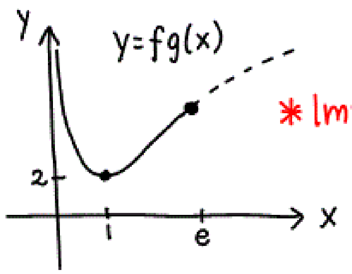
Given that $\mathrm{f}(x) = x^2 + 2, x \in \mathbb{R}$ and $\mathrm{g}(x) = \ln x,\ 0 < x \le e$. Find the range of $\mathrm{fg}$.

**Method 1:**

Sketch the graph of fg($x$) based on the domain of fg and find range based on the graph.

$fg(x) = f(\ln x) = (\ln x)^2 + 2$

$D_{fg} = D_g = (0, e]$




* Important to sketch $y = fg(x)$ according to $D_{fg}$.

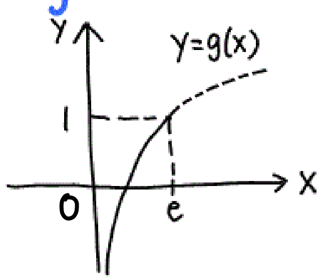
$\therefore R_{fg} = [2, \infty)$

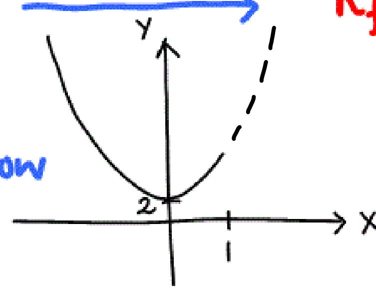
**Method 2:** Use $\mathrm{R_g}$ as domain of f and find the corresponding range of f using the graph of f.

$D_g \xrightarrow{g} R_g = (-\infty, 1]$ $\xrightarrow{f}$ $R_{fg} = [2, \infty)$

$=$ output of $g$

$=$ input for $f$ now




Sketch $y = f(x)$ where $x \in (-\infty, 1]$

The range obtained will be $R_{fg}$. :)