Solutions to Self-Practice on Differentiation

1. $x^3y^2+x^2y^3=1$ ---(1)

Differentiate with respect to x:

$$x^3\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)+y^2\left(3x^2\right)+x^2\left(3y^2\frac{\mathrm{d}y}{\mathrm{d}x}\right)+y^3\left(2x\right)=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(2x^3y+3x^2y^2\right)=-\left(2y^3x+3x^2y^2\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{xy^2\left(2y+3x\right)}{x^2y\left(2x+3y\right)}$$

For tangent parallel to y-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}\to\pm\infty$.

So let denominator of be 0.

$$x^2y(2x+3y)=0$$

$$\Rightarrow x=-\frac{3}{2}y \quad \text{or } x=0 \quad \text{or } y=0$$

If x or y is 0 then $x^3y^2+x^2y^3=1$ will be 0 = 1. So $x\neq 0, y\neq 0$.

Therefore $x=-\frac{3}{2}y$ --- (2)

Substitute (2) into (1) eqn of curve:

$$\left(-\frac{3}{2}y\right)^3y^2+\left(-\frac{3}{2}y\right)^2y^3=1$$

$$-\frac{27}{8}y^5+\frac{9}{4}y^5=1$$

$$-\frac{9}{8}y^5=1$$

$$y=\sqrt[5]{-\frac{8}{9}}$$

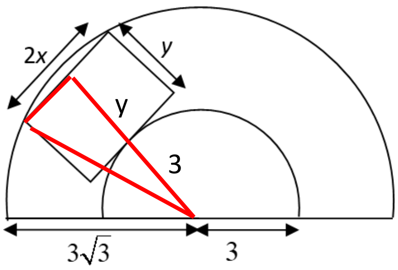
$$y=-\sqrt[5]{\frac{8}{9}}$$

$$x=-\frac{3}{2}\left(-\sqrt[5]{\frac{8}{9}}\right)=\frac{3}{2}\sqrt[5]{\frac{8}{9}}$$

$$\left(\frac{3}{2}\sqrt[5]{\frac{8}{9}},-\sqrt[5]{\frac{8}{9}}\right)$$

Tangent is // to y-axis means tangent is a vertical line so the equation of tangent is $x=\frac{3}{2}\sqrt[5]{\frac{8}{9}}$.

2(i)Pythagoras theorem:



$$(3+y)^2 + x^2 = \left(3\sqrt{3}\right)^2$$
$$9+6y+y^2+x^2 = 27$$
$$x^2 = 18-6y-y^2$$

Area of the sheet, $A = 2xy$

$$= 2y\sqrt{18-6y-y^2}$$
$$= 2\sqrt{18y^2-6y^3-y^4}$$

(ii)

$$\frac{dA}{dy} = 2\frac{1}{2\sqrt{18y^2-6y^3-y^4}}\left(36y-18y^2-4y^3\right) = \frac{2y\left(18-9y-2y^2\right)}{\sqrt{18y^2-6y^3-y^4}}$$

[OR $\quad A^2 = 4\left(18y^2-6y^3-y^4\right)$

$$2A\frac{dA}{dy} = 4\left(36y-18y^2-4y^3\right)$$
$$A\frac{dA}{dy} = 4y\left(18-9y-2y^2\right)$$]

For maximum *A*, $\quad$ *A* must be stationary so $\frac{dA}{dy} = 0$

$$\frac{2y\left(18-9y-2y^2\right)}{\sqrt{18y^2-6y^3-y^4}} = 0$$
$$2y\left(18-9y-2y^2\right) = 0$$
$$2y^2+9y-18 = 0 \text{ or } y=0 \text{ (reject as } y>0)$$
$$(2y-3)(y+6) = 0$$
$$y = \frac{3}{2} \text{ or } y=-6 \text{ (reject as } y>0)$$

y	1.4	$\left(\frac{3}{2}\right)$	1.6
$\frac{dA}{dy}$	$1.08 > 0$	0	$-1.26 < 0$

A is maximum when $y = \frac{3}{2}$

When $y = \frac{3}{2}$,

Maximum $A == 2\sqrt{18\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^4} = \frac{9\sqrt{3}}{2}$ =7.79 m^2

3(i) Observe that the arc length of the sector in Diagram 1 is also the circumference of the circle in Diagram 2. Formula for Arc Length is $r\theta$ where θ is in radians.

$a\theta = 2\pi r$, i.e., $r = \dfrac{a\theta}{2\pi}$.

(ii)

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\left(\pi r^2\right)\sqrt{a^2 - r^2}$$
$$= \frac{1}{3}\left[\pi\left(\frac{a\theta}{2\pi}\right)^2\right]\sqrt{a^2 - \left(\frac{a\theta}{2\pi}\right)^2}$$
$$= \frac{a^2\theta^2}{12\pi}\sqrt{a^2\left(1 - \frac{\theta^2}{4\pi^2}\right)}$$
$$= \frac{a^2\theta^2}{12\pi}\sqrt{a^2\left(\frac{4\pi^2 - \theta^2}{4\pi^2}\right)}$$
$$V^2 = \left(\frac{a^2\theta^2}{12\pi}\right)^2 a^2\left(\frac{4\pi^2 - \theta^2}{4\pi^2}\right)$$
$$V^2 = \frac{a^6\theta^4}{576\pi^4}\left(4\pi^2 - \theta^2\right) \text{ (Shown)}$$

(iii)

when $a = 2$, $V^2 = \dfrac{2^6\theta^4}{576\pi^4}\left(4\pi^2 - \theta^2\right)$

$$V^2 = \frac{1}{9\pi^4}(4\pi^2\theta^4 - \theta^6)$$

Differentiating $V^2 = \dfrac{a^6\theta^4}{576\pi^4}\left(4\pi^2 - \theta^2\right)$ with respect to θ

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(V^2\right) = \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{1}{9\pi^4}(4\pi^2\theta^4 - \theta^6)\right]$$
$$\frac{\mathrm{d}}{\mathrm{d}V}\left(V^2\right)\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{1}{9\pi^4}(4\pi^2\theta^4 - \theta^6)\right]$$
$$2V\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{1}{9\pi^4}\left(16\pi^2\theta^3 - 6\theta^5\right)$$

Let $\dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0$,

$16\pi^2\theta^3 - 6\theta^5 = 0$

$2\theta^3\left(8\pi^2 - 3\theta^2\right) = 0$, i.e.,

$\theta = 0$ (N.A.) or $\theta = \pm\sqrt{\dfrac{8\pi^2}{3}}$ (reject –ve).

Thus, $\theta = \sqrt{\dfrac{8\pi^2}{3}}$ and $V = \dfrac{8\pi^2}{3}\left(\dfrac{1}{3\pi^2}\right)\sqrt{\left(4\pi^2 - \dfrac{8\pi^2}{3}\right)} = \dfrac{8}{9}\sqrt{\dfrac{4\pi^2}{3}} = \dfrac{16\sqrt{3}}{27}\pi$.

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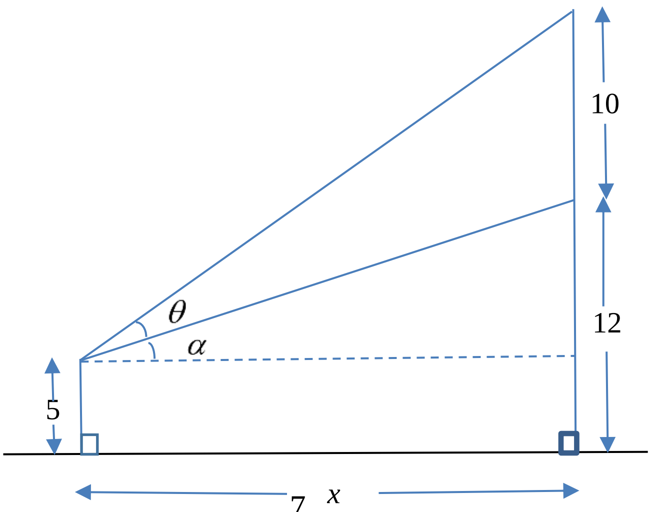

From diagram, $\tan\alpha = \dfrac{7}{x}$

and $\tan\left(\theta+\alpha\right) = \dfrac{17}{x}$

$$\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha} = \frac{17}{x}$$

$$\frac{\tan\theta+\frac{7}{x}}{1-\tan\theta(\frac{7}{x})} = \frac{17}{x}$$

$$x\tan\theta+7 = 17-\frac{119}{x}\tan\theta$$

$$\tan\theta\left(x+\frac{119}{x}\right) = 10$$

$$\tan\theta = \frac{10}{\left(x+\frac{119}{x}\right)}$$

$$= \frac{10x}{x^2+119}$$

Differentiating wrt x,

$$\frac{d}{dx}\left(\tan\theta\right) = \frac{d}{dx}\left(\frac{10x}{x^2+119}\right)$$

$$\frac{d}{d\theta}\left(\tan\theta\right)\frac{d\theta}{dx} = \frac{d}{dx}\left(\frac{10x}{x^2+119}\right)$$

$$\sec^2\theta\frac{d\theta}{dx} = \frac{(119+x^2)(10)-10x(2x)}{(119+x^2)^2}$$

$$\sec^2\theta\frac{d\theta}{dx} = \frac{1190-10x^2}{(119+x^2)^2}$$

For maximum angle θ, θ is stationary, so

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = 0$$

$$\sec^2\theta.(0) = \frac{1190 - 10x^2}{(119 + x^2)^2}$$

$$1190 - 10x^2 = 0$$

$$\Rightarrow 10x^2 = 1190$$

Since $x > 0$, $x = \sqrt{119}$ (exact)

Since $\sec^2\theta > 0$, $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{1190 - 10x^2}{(119 + x^2)^2} \cdot \dfrac{1}{\sec^2\theta}$

Sign of $\dfrac{\mathrm{d}\theta}{\mathrm{d}x}$ = sign of $\dfrac{1190 - 10x^2}{(119 + x^2)^2}$ = sign of $\left(1190 - 10x^2\right)$

	10.8	$\left(\sqrt{119}\right)$	11
Sign of $1190 - 10x^2$	$23.6 > 0$	0	$-20 < 0$
slope	/	—	\

Therefore θ is maximum when $x = \sqrt{119}$.