Completing the Squares and Quadratic Polynomials

Basic Notes:

In completing the squares, we will need to perform the following steps:

Step 1: Make sure that the coefficient of x^2 is 1 after factorizing suitable constant;

For example, $ax^2+bx+c=a(x^2+\frac{b}{a}x+\frac{c}{a})$.

Step 2: Divide the coefficient of x (i.e., $\frac{b}{a}$) by 2 to obtain $\frac{b}{2a}$;

Step 3: Add a copy of $\left(\frac{b}{2a}\right)^2$ and subtract a copy of $\left(\frac{b}{2a}\right)^2$ and finally convert the resulting expression into the form $a\left[(x+p)^2\right]+q$.

$$\begin{aligned}ax^2+bx+c=a(x^2+\frac{b}{a}x+\frac{c}{a})&=a(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2)\\&=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]\\&=a(x+\frac{b}{2a})^2-\frac{b^2-4ac}{4a}.\end{aligned}$$

Quadratic Equations

By the above identity, the quadratic equation $ax^2+bx+c=0$ $(a\neq 0)$ becomes

$$a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]=0.$$

From this, we obtain the roots to be $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

The nature of the roots of the quadratic equation $ax^2+bx+c=0$ $(a\neq 0)$ depends on the value b^2-4ac, which is called the discriminant of the equation.

The roots are categorized as follows:

(1) If $b^2-4ac>0$, the two roots are real and distinct.

(2) If $b^2-4ac=0$, the two roots are real and equal.

(3) If $b^2-4ac<0$, the quadratic equation has no real roots

We conclude that the quadratic equation has real roots if and only if $b^2-4ac\geq 0$.

Worksheet: Completing the Squares and Quadratic Polynomials

1. Complete the squares of the followings and state the maximum or minimum value of y:

(i) $y = x^2 + 2x - 3$
(ii) $y = -x^2 + 3x - 4$
(iii) $y = 3x^2 + 6x + 9$
(iv) $y = 2x^2 - 8x + 8$

Sketch the graphs for (i) and (ii).

2. Find a, b and c which satisfy the identity

$$3x^2 + 4x - 1 \equiv a(x-1)(x-2) + b(x-1) + c.$$

3. Find constants a, b and c such that $x^2 - 7x + 9 \equiv a(x-b)^2 + c.$

4. Express $1 + x - 2x^2$ in the form $b - c(x-a)^2$ and find the maximum value of the expression.

5. Find the possible values of k if $x^2 + (k-3)x + 4 = 0$ has
(i) real and distinct roots
(ii) equal roots.

6. Find the range of values of k for which the equation $x^2 + 2(k-1)x + k - 1 = 0$ has no real roots.

Answers:

(1) (i) $(x+1)^2 - 4;$ min $y = -4;$ (ii) $-(x-\frac{3}{2})^2 - \frac{7}{4};$ max $y = -\frac{7}{4};$
(iii) $3(x+1)^2 + 6;$ min $y = 6;$ (iv) $2(x-2)^2;$ min $y = 0.$
(2) $a = 3, b = 13, c = 6$; (3) $a = 1, b = \frac{7}{2}, c = -\frac{13}{4}$; (4) $\frac{9}{8} - 2(x - \frac{1}{4})^2, \frac{9}{8}$;
(5) (i) $k > 7$ or $k < -1$ (ii) $k = 7, -1.$; (6) $1 < k < 2.$