Revision Notes for Vectors
(Formulae and Procedures you must know)

(1) For two vectors **a** and **b**,

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle in the interval $[0,\pi]$ between vectors **a** and **b**.

Remarks:

(i) Note that (i) $\mathbf{a}.\mathbf{a} = |\mathbf{a}|^2$ and (ii) $(\mathbf{b}-\mathbf{a}).(\mathbf{b}-\mathbf{a}) = |\mathbf{b}-\mathbf{a}|^2$.

(ii) If $\mathbf{a}.\mathbf{b} = 0,$ this implies that the vectors **a** and **b** are perpendicular

<u>Example to illustrate how to apply Remark (i) above:</u>

Question: Given that $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 2$ and the angle between **a** and **b** is 60°, find $|2\mathbf{a}+3\mathbf{b}|$.

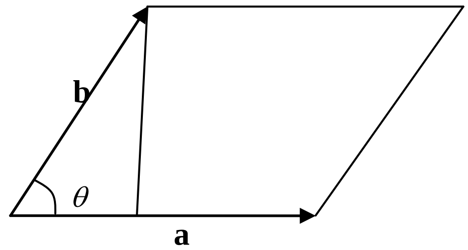
[Solution]

$$|2\mathbf{a}+3\mathbf{b}|^2$$
$$= (2\mathbf{a}+3\mathbf{b})\cdot(2\mathbf{a}+3\mathbf{b})$$
$$= 4\mathbf{a}\cdot\mathbf{a} + 6\mathbf{a}\cdot\mathbf{b} + 6\mathbf{b}\cdot\mathbf{a} + 9\mathbf{b}\cdot\mathbf{b}$$
$$= 4|\mathbf{a}|^2 + 12\mathbf{a}\cdot\mathbf{b} + 9|\mathbf{b}|^2$$
$$= 4|\mathbf{a}|^2 + 12|\mathbf{a}||\mathbf{b}|\cos 60^\circ + 9|\mathbf{b}|^2$$
$$= 4(1) + 12(1)(2)(\frac{1}{2}) + 9(2)^2$$
$$= 52$$

Hence, $|2\mathbf{a}+3\mathbf{b}| = \sqrt{52} = 2\sqrt{13}.$

(2) $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

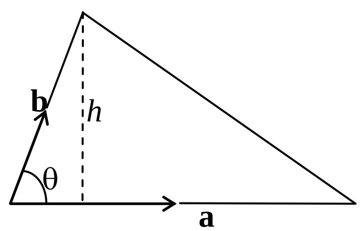
(3) <u>Area of a parallelogram</u>



Area of the parallelogram with vectors **a** and **b** representing its two adjacent sides is $|\mathbf{a} \times \mathbf{b}|$.

(4) <u>Area of a Triangle</u>

The area of a triangle with vectors **a** and **b** representing its two adjacent sides is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.



(5) <u>Parallel lines, Perpendicular lines, skew lines and intersecting lines</u>

(a) 2 lines are parallel if the direction vectors of the two lines are parallel to each other.

(b) 2 lines intersect if we can find a point common to the two lines. You will have to solve the equations of the two lines simultaneously for the point of intersection.

(c) 2 lines are perpendicular if the direction vectors are perpendicular to each other. You will have to show that the dot product of the direction vectors of the 2 lines is zero.

(d) 2 lines are called skew lines if the two lines do not meet and the 2 lines are not parallel to each other. To prove skew lines, you have to prove that (1) the lines are not parallel and (2) the two lines do not intersect.

(6) Length of projection of a vector onto a line with direction vector ***d***

The length of projection of the vector $\overrightarrow{PQ}$ onto a line l with direction vector $\boldsymbol{d}$ is $\left|\overrightarrow{PQ}.\hat{\mathbf{d}}\right|$ where $\hat{\mathbf{d}}$ is the unit vector.

Remark:
$\left|\mathbf{a}\cdot\hat{\mathbf{b}}\right|$ is described as the length of projection of vector **a** onto vector **b**.

$\left|\mathbf{b}\cdot\hat{\mathbf{a}}\right|$ is described as the length of projection of vector **b** onto vector **a**.

Do look out for which vector is the unit vector before you describe.

(7) Angle between 2 lines

Let l_1 be the line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}_1$
Let l_2 be the line with equation $\mathbf{r} = \mathbf{b} + \mu\mathbf{d}_2$
The **acute** angle between these two lines, θ, is given by

$$\cos\theta = \frac{|\mathbf{d}_1\cdot\mathbf{d}_2|}{|\mathbf{d}_1||\mathbf{d}_2|}.$$

$$\Rightarrow \theta = \cos^{-1}\frac{|\mathbf{d}_1\cdot\mathbf{d}_2|}{|\mathbf{d}_1||\mathbf{d}_2|}.$$

(8) Shortest distance between 2 parallel lines

Let l_1 and l_2 be 2 parallel lines with equations $\mathbf{r} = \overrightarrow{OA} + \lambda\mathbf{d}_1$ and $\mathbf{r} = \overrightarrow{OB} + \mu\mathbf{d}_1$ respectively.

The shortest distance between these 2 lines, h, is given by

$$h = \left|\overrightarrow{AB}\times\hat{\mathbf{d}}_1\right|.$$

where $\hat{\mathbf{d}}_1$ is the unit vector.

(9) <u>Shortest distance from a point to a line</u>

Let l_1 be the line with equation $\mathbf{r} = \overrightarrow{OA} + \lambda\mathbf{d}_1$. The shortest distance, *h*, from a point *C* with position vector $\overrightarrow{OC}$ to the line l_1 is given by

$$h = \left|\overrightarrow{AC} \times \hat{\mathbf{d}}_1\right|$$

where $\hat{\mathbf{d}}_1$ is the unit vector.

(10) <u>Angle between 2 planes</u>

Let Π_1 and Π_2 be 2 planes with equations $\mathbf{r} \cdot \mathbf{n}_1 = p_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = p_2$ respectively.

The **acute** angle between these 2 planes, θ, is given by

$$\cos\theta = \frac{\left|\mathbf{n}_1 \cdot \mathbf{n}_2\right|}{\left|\mathbf{n}_1\right|\left|\mathbf{n}_2\right|}$$

$$\Rightarrow \theta = \cos^{-1}\frac{\left|\mathbf{n}_1 \cdot \mathbf{n}_2\right|}{\left|\mathbf{n}_1\right|\left|\mathbf{n}_2\right|}.$$

(11) <u>Angle between a line and a plane</u>

Let l_1 be the line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}_1$ and Π_1 be the plane with equation $\mathbf{r} \cdot \mathbf{n}_1 = p_1$.

The acute angle between the line and the plane, θ, is given by

$$\sin\theta = \frac{\left|\mathbf{d}_1 \cdot \mathbf{n}_1\right|}{\left|\mathbf{d}_1\right|\left|\mathbf{n}_1\right|}$$

$$\Rightarrow \theta = \sin^{-1}\frac{\left|\mathbf{d}_1 \cdot \mathbf{n}_1\right|}{\left|\mathbf{d}_1\right|\left|\mathbf{n}_1\right|}.$$

(12) Shortest distance between 2 parallel planes

Let Π_1 and Π_2 be 2 planes with equations $\mathbf{r}\cdot\mathbf{n}_1 = p_1$ and $\mathbf{r}\cdot\mathbf{n}_1 = p_2$ respectively.

[Method]
Find a point, *A*, on the plane Π_1 by trial and error.
Find a point, *B*, on the plane Π_2 by trial and error.

The shortest distance between the 2 planes, *h*, is given by,

$$h = \left|\overrightarrow{AB}\cdot\hat{\mathbf{n}}_1\right|$$

where $\hat{\mathbf{n}}_1$ is the unit vector.

(13) Perpendicular distance from a point to a plane

To find the perpendicular or shortest distance from a point *A* to a plane, we can use the following 2 methods:

[Method 1]

Formulate the equation of the line passing through *A* and perpendicular to the plane. Then we solve the equation of this line with the plane simultaneously to get the position vector of the foot of perpendicular, say *F*, from *A* to the plane. We will then find the vector $\overrightarrow{AF}$ and the perpendicular distance is given by $\left|\overrightarrow{AF}\right|$.

[Method 2]

Find a point, say *B*, on the plane by trial and error. Then we will compute the vector $\overrightarrow{BA}$ and the shortest distance from *A* to the plane is given by $\left|\overrightarrow{BA}\cdot\hat{n}\right|$ where $\hat{n}$ is the unit vector perpendicular to the plane.

(14) <u>Shortest or Perpendicular distance from the origin to a plane</u>

If a plane has equation $\mathrm{r} \cdot n = d$, then the perpendicular distance from the origin O to the plane is given by $\frac{|d|}{|n|}$.

For example, given a plane with equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -5,$ the perpendicular distance from the origin to this plane is $\frac{|-5|}{\left|\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right|} = \frac{5}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{5}{\sqrt{14}}.$

(15) <u>Length of projection of a vector onto a plane</u>

The length of projection of a vector $\overrightarrow{AB}$ onto a plane with $\mathrm{r} \cdot n = d$ is given by $\left|\overrightarrow{AB} \times \hat{n}\right|$ where $\hat{n}$ is the unit vector perpendicular to the plane.

(16) <u>Others</u>

Do revise

(1) how to find position vector of foot of perpendicular from a point to a line
(2) finding vector equation of line of intersection of two non-parallel planes