Solutions to Integration :)

1. (i) $\int \cos^2 x \, dx = \int \frac{1+\cos 2x}{2} \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

(ii) $\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

2. (i) $\tan x + c$ (ii) $\int \sec^2 x - 1 \, dx = \tan x - x + c$

3 (i) $\ln|\sec x + \tan x| + c$ (ii) $\ln|\sec x| + c$ ← remember modulus for ln

(iii) $\frac{d}{dx}(\sec x) = \tan x \sec x$ (MF 26) $\Rightarrow \int \tan x \sec x \, dx = \sec x + c$

4. (i) $\int (\sin x)^2 \cos x \, dx =$ " $\int (f(x))^2 f'(x) \, dx$ " $= \frac{(\sin x)^3}{3} + c$

(ii) $\int (\tan x)^3 \sec^2 x \, dx = \frac{(\tan x)^4}{4} + c$

5. MF 26 : $2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2} = \sin P + \sin Q$

let $\frac{P+Q}{2} = 4x$ ①, $\frac{P-Q}{2} = 3x$ ② $\Rightarrow$ Solve : ① + ② $\Rightarrow P = 4x + 3x = 7x$

① − ② $\Rightarrow Q = 4x - 3x = x$

$\therefore \int \sin 4x \cos 3x \, dx = \frac{1}{2}\int \sin 7x + \sin x \, dx = \frac{1}{2}\left[-\frac{\cos 7x}{7} - \cos x\right] + c$

6. (i) $\int 1 + \frac{1}{x^2-1} \, dx = x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c$ [Note: Improper frac., modulus sign for ln]

(ii) $\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x-1)$

sub $x=1$, $A = \frac{1}{2}$ sub $x=0$, $C = -\frac{1}{2}$ sub $x=2$, $B = -\frac{1}{2}$

$\int \frac{1}{(x-1)(x^2+1)} dx = \int \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}x - \frac{1}{2}}{x^2+1} \, dx = \frac{1}{2}\ln|x-1| + \left(-\frac{1}{2}\right)\int \frac{x}{x^2+1} \, dx$

$- \frac{1}{2}\int \frac{1}{x^2+1} \, dx$

$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\tan^{-1} x + c$

7. (i) $\int \frac{1}{\sqrt{4-9x^2}} \, dx = \int \frac{1}{\sqrt{2^2-(3x)^2}} \, dx = \left(\frac{1}{3}\right)\left(\sin^{-1}\frac{3x}{2}\right) + c$ ☆ Remember $\frac{1}{3}$ if you replace x by 3x

(ii) $\int \frac{1}{\sqrt{9-(x+1)^2}} \, dx = \sin^{-1}\left(\frac{x+1}{3}\right) + c$

8. (i) $\int \frac{1}{2^2+(3x)^2} \, dx = \frac{1}{3}\left(\frac{1}{2}\tan^{-1}\frac{3x}{2}\right) + c = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + c$

(ii) $\int \frac{1}{9-(x+1)^2} \, dx = \frac{1}{2(3)}\ln\left|\frac{3+(x+1)}{3-(x+1)}\right| + c = \frac{1}{6}\ln\left|\frac{4+x}{2-x}\right| + c$

9 (i) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$\Rightarrow \int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = 2\int \frac{1}{2\sqrt{x}} e^{\sqrt{x}} \, dx = 2e^{\sqrt{x}} + c$ [Use $\int f'(x) e^{f(x)} dx$]

(ii) $\int x(1-x^2)^{-\frac{1}{2}} \, dx = -\frac{1}{2}\int -2x(1-x^2)^{-\frac{1}{2}} \, dx = -\frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -\sqrt{1-x^2} + c$

10 (i) $\int \frac{x}{\sqrt{4-x^2}} + \frac{3}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\int (-2x)(4-x^2)^{-\frac{1}{2}}\,dx + 3\int \frac{1}{\sqrt{2^2-x^2}}\,dx$

$= -\frac{1}{2}\,\frac{(4-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + 3\sin^{-1}\frac{x}{2} + c$

$= -\sqrt{4-x^2} + 3\sin^{-1}\left(\frac{x}{2}\right) + c$

(ii) $\int \frac{x}{x^2-2x+3}\,dx = \int \frac{A(2x-2)+B}{x^2-2x+3}\,dx$

By compare coeff: $x = A(2x-2) + B$

$x \Rightarrow x = 2Ax$

$A = \frac{1}{2}$

constant $\Rightarrow 0 = -2A + B$

$B = 2A = 1$

$= \int \frac{\frac{1}{2}(2x-2)}{x^2-2x+3}\,dx + \int \frac{1}{(x-1)^2+2}\,dx$

$= \frac{1}{2}\ln|x^2-2x+3| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + c$

11 (i) let $u = (\ln x)^2$ [Note: $(\ln x)^2 \neq 2\ln x$] $v' = 1$

$u' = 2(\ln x)\frac{1}{x}$ $v = x$

$\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2\ln x\,dx$ $u = \ln x$, $v' = 1$, $u' = \frac{1}{x}$, $v = x$

$= x(\ln x)^2 - 2\left[x\ln x - \int 1\,dx\right]$

$= x(\ln x)^2 - 2x\ln x + 2x + c$

(ii) $\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx$ $u = \sin x$, $v' = e^x$, $u' = \cos x$, $v = e^x$

$= e^x\sin x - \left[e^x\cos x - \int -e^x\sin x\,dx\right]$ $u = \cos x$, $v' = e^x$, $u' = -\sin x$, $v = e^x$

$= e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$

$2\int e^x\sin x\,dx = e^x\sin x - e^x\cos x + c$

$\int e^x\sin x\,dx = \frac{1}{2}(e^x\sin x - e^x\cos x) + c'$

(iii) $\int \sin^{-1}2x\,dx = x\sin^{-1}2x - \int \frac{2x}{\sqrt{1-4x^2}}\,dx$ $u = \sin^{-1}2x$, $v' = 1$, $u' = \frac{2}{\sqrt{1-4x^2}}$, $v = x$

$= x\sin^{-1}2x - \left(-\frac{1}{4}\right)\int(-8x)(1-4x^2)^{-\frac{1}{2}}\,dx$

$= x\sin^{-1}2x + \frac{1}{4}\,\frac{(1-4x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c = x\sin^{-1}2x + \frac{1}{2}\sqrt{1-4x^2} + c$

12. $\int x^2(xe^{x^2})\,dx = \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}\,dx$ $u = x^2$, $v' = xe^{x^2}$, $u' = 2x$, $v = \frac{1}{2}e^{x^2}$

$= \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + c$

13. $\int a^x\,dx = \int e^{\ln a^x}\,dx = \int e^{x\ln a}\,dx = \int e^{(\ln a)x}\,dx = \frac{1}{\ln a}e^{x\ln a} + c$

$= \frac{1}{\ln a}a^x + c$