SUMMARY NOTES ON INTEGRATION TECHNIQUES

(A) Basic Properties of Indefinite Integral:

Let f and g be two functions. Then

(i) $\int k\,\mathrm{f}(x)\mathrm{d}x = k\int \mathrm{f}(x)\mathrm{d}x$, where k is a real constant.

(ii) $\int [\mathrm{f}(x) \pm \mathrm{g}(x)]\,\mathrm{d}x = \int \mathrm{f}(x)\,\mathrm{d}x \ \pm \ \int \mathrm{g}(x)\,\mathrm{d}x$.

(B) Basic properties of definite integral:

(i) $\int_a^a \mathrm{f}(x)\,\mathrm{d}x = 0$,

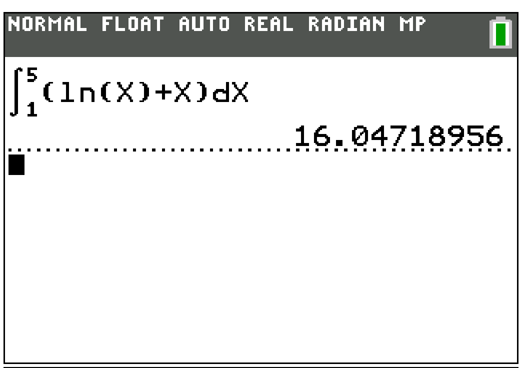
(ii) $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = -\int_b^a \mathrm{f}(x)\,\mathrm{d}x$,

(iii) $\int_a^b k\,\mathrm{f}(x)\,\mathrm{d}x = k\int_a^b \mathrm{f}(x)\,\mathrm{d}x$, where k is a real constant.

(iv) $\int_a^c \mathrm{f}(x)\,\mathrm{d}x = \int_a^b \mathrm{f}(x)\,\mathrm{d}x + \int_b^c \mathrm{f}(x)\,\mathrm{d}x$, where $a < b < c$.

(C) Revise your GC keystroke

By using a graphing calculator, find the value of $\int_1^5 \ln x + x \ \mathrm{d}x$, giving your answer correct to 1 decimal place.



Hence, answer is 16.0 to 1 decimal place.

(D) Integration by Standard Forms

(1) Let n be a constant $(n \neq -1)$.

(i) $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C$

(ii) $\int (ax+b)^n \, dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$

(iii) $\int f'(x)\left[f(x)\right]^n dx = \dfrac{\left[f(x)\right]^{n+1}}{n+1} + C$

(2)

(i) $\int \dfrac{1}{x} \, dx = \ln|x| + C$

(ii) $\int \dfrac{1}{f(x)} f'(x) \, dx = \ln\left| f(x) \right| + C$

(3)

(i) $\int e^x \, dx = e^x + C$

(ii) $\int f'(x) \, e^{f(x)} \, dx = e^{f(x)} + C$

Integration involving Trigonometry:

(1) To integrate $\int \sin^2 x \, dx$, $\int \cos^2 x \, dx$ **or** $\int \tan^2 x \, dx$, you must use the identities:

$\cos 2x = 1 - 2\sin^2 x$

$\cos 2x = 2\cos^2 x - 1$

$1 + \tan^2 x = \sec^2 x$

(2) To integrate $\int \cos 4x \cos 2x \, dx$, you must use factor formula (found in MF15) and integrate piece by piece.

$$\int \cos 4x \cos 2x \, dx = \frac{1}{2}\int \cos 6x + \cos 2x \, dx = \frac{1}{12}\sin 6x + \frac{1}{4}\sin 2x + c.$$

(i) $\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$; $\int \frac{1}{(x+b)^2+a^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right)+C$

(ii) $\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$; $\int \frac{1}{(x+b)^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{(x+b)-a}{(x+b)+a}\right|+C$

(iii) $\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C$; $\int \frac{1}{a^2-(x+b)^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{a+(x+b)}{a-(x+b)}\right|+C$

(iv) $\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right)+C$,

(v) $\int \frac{1}{\sqrt{a^2-(x+b)^2}}\,\mathrm{d}x = \sin^{-1}\left(\frac{x+b}{a}\right)+C$

Remarks:

(a) Sometimes you will need to complete the squares before applying the formulae above.

(b) If you see an improper fraction, you might need to do long division before applying formula under standard forms.

For example, $\int \frac{x^2}{x^2+1}\,\mathrm{d}x = \int 1-\frac{1}{x^2+1}\,\mathrm{d}x = x-\tan^{-1}x+c.$

(E) Integration by partial fractions

$$\int \frac{x}{(x+1)(x+2)(x+3)}\,\mathrm{d}x$$
$$=\int (-\frac{1}{2})(\frac{1}{x+1})+2(\frac{1}{x+2})+(-\frac{3}{2})(\frac{1}{x+3})\,\mathrm{d}x$$
$$=-\frac{1}{2}\ln|x+1|+2\ln|x+2|-\frac{3}{2}\ln|x+3|+c$$

(F) Integration by Substitution

Example: Using the substitution $x = a\sin\theta$, where a is a positive constant, show that

$$\int_{\frac{a}{2}}^{a} \sqrt{a^2 - x^2}\ \mathrm{d}x = \frac{a^2}{24}\left(4\pi - 3\sqrt{3}\right)$$

[Solution]

$$x = a\sin\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = a\cos\theta$$

$$\int_{\frac{a}{2}}^{a} \sqrt{a^2 - x^2}\ \mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{a^2 - \left(a\sin\theta\right)^2}\ \left(a\cos\theta\right)\mathrm{d}\theta$$

$$= a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\cos^2\theta\right)\mathrm{d}\theta$$

$$= a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\left(\cos 2\theta + 1\right)}{2}\mathrm{d}\theta$$

$$= \frac{a^2}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{2} - \frac{1}{2}\sin\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{a^2}{2}\left[\frac{\pi}{3} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= -\frac{1}{8}\sqrt{3}a^2 + \frac{\pi}{6}a^2 = \frac{a^2}{24}\left(4\pi - 3\sqrt{3}\right)$$

(G) Integration by Parts

Formula: $\int u\, v' \mathrm{d}x \quad = \quad u\, v - \int u'\, v\, \mathrm{d}x$

$$\int_a^b uv' \mathrm{d}x = \left[uv\right]_a^b - \int_a^b u' v \; \mathrm{d}x$$

Example: Find $\int x \sin x \; \mathrm{d}x$.

[Solution]:

$u = x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$

$v = -\cos x$

$$\int x \sin x \; \mathrm{d}x = -x \cos x + \int \cos x \; \mathrm{d}x = -x \cos x + \sin x + c.$$

Miscellaneous Examples:

Example 1

[Key Technique: Writing $6+2x=A(-2x-4)+B$ where $\dfrac{\mathrm{d}}{\mathrm{d}x}(1-4x-x^2)=-2x-4$ **]**

Find $\displaystyle\int \frac{6+2x}{\sqrt{1-4x-x^2}}\ \mathrm{d}x$.

[Solution]:

$$\int \frac{6+2x}{\sqrt{1-4x-x^2}}\ \mathrm{d}x=\int \frac{2-\left(-4-2x\right)}{\sqrt{1-4x-x^2}}\ \mathrm{d}x$$

$$=\int \frac{2}{\sqrt{1-4x-x^2}}\ \mathrm{d}x-\int \frac{\left(-4-2x\right)}{\sqrt{1-4x-x^2}}\ \mathrm{d}x$$

$$=\int \frac{2}{\sqrt{5-(x+2)^2}}\ \mathrm{d}x-\int \frac{-4-2x}{\sqrt{1-4x-x^2}}\ \mathrm{d}x$$

$$=\int \frac{2}{\sqrt{5-(x+2)^2}}\ \mathrm{d}x-\int (-4-2x)(1-4x-x^2)^{-\frac{1}{2}}\mathrm{d}x$$

$$=\int \frac{2}{\sqrt{5-(x+2)^2}}\ \mathrm{d}x-\frac{(1-4x-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$=2\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)-2\sqrt{1-4x-x^2}+c$$

Example 2:
[Integration involving absolute value which requires us to break up the domain into relevant intervals]

Find $\int_0^2 \left|x^2+2x-3\right| dx$.

[Solution]

Solving $x^2+2x-3<0 \Rightarrow (x-1)(x+3)<0 \Rightarrow -3<x<1.$

$\therefore$

For $0<x<1,$ we have $x^2+2x-3<0$ and hence $\left|x^2+2x-3\right| = -\left(x^2+2x-3\right) = -x^2-2x+3.$

For $1<x<2,$ we have $x^2+2x-3>0$ and hence $\left|x^2+2x-3\right| = x^2+2x-3.$

$$\begin{aligned}\int_0^2 \left|x^2+2x-3\right| dx &= \int_0^1 -x^2-2x+3\ dx + \int_1^2 x^2+2x-3\ dx \\ &= \left[-\frac{1}{3}x^3-x^2+3x\right]_0^1 + \left[\frac{1}{3}x^3+x^2-3x\right]_1^2 \\ &= 4\end{aligned}$$

Example 3 [Integration by parts 2 times]

Find $\int e^{2x} \cos 4x \, dx$.

[Solution]:

$$\int e^{2x} \cos 4x \, dx = \frac{1}{2} e^{2x} \cos 4x - \frac{1}{2} \int e^{2x} (-4 \sin 4x) \, dx$$

$$= \frac{1}{2} e^{2x} \cos 4x + 2 \int e^{2x} \sin 4x \, dx$$

$$= \frac{1}{2} e^{2x} \cos 4x + 2 \left(\frac{1}{2} e^{2x} \sin 4x - \frac{1}{2} \int e^{2x} (4 \cos 4x) \, dx \right)$$

$$= \frac{1}{2} e^{2x} \cos 4x + e^{2x} \sin 4x - 4 \int e^{2x} \cos 4x \, dx$$

$$5 \int e^{2x} \cos 4x \, dx = \frac{1}{2} e^{2x} (\cos 4x + 2 \sin 4x) + c'$$

$$\int e^{2x} \cos 4x \, dx = \frac{1}{10} e^{2x} (\cos 4x + 2 \sin 4x) + c$$